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# SOME METHODS FOR SOLVING EQUILIBRIUM PROBLEMS OVER FIXED POINT SETS

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**SUMMARY OF DOCTORAL THESIS**

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This thesis has been completed at Thang Long University, based on the research results of the author and colleagues.

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# contents

# Introduction

## Overview of research situation

After more than half a century of formation and development, the theory of equilibrium problems over fixed point sets has gradually con- firmed its role as well as the development in optimization theory, applied mathematics and realistic mathematical models.

Nikaido H. and Isoda K. first introduced the equilibrium problem in 1955. After that, Ky Fan (1972) reviewed this model that is the form of a minimax inequality. Then, existence of solution of the problem is shown under the convex and compact condition of the set C and the function f is quasi-convex on $C$ . This result is also extended by Brezis H. and colleagues in 1987. In 1992, Muu L.D. and Oettli W. introduced first the equilibrium problem and a new algorithm to solve it, where the bifunction f is monotone. After the research of Blum E. and Oettli W., in 1994 was published, the equilibrium problem has attracted the attention of many researchers.

Although the equilibrium problem has a fairly simple form but it con- tains many important classes of problems in many other fields such as optimization problem, saddle point problem, inequality variational in- equality problem, fixed point problem, Nash equilibrium problem. From the results of the individual problems mentioned above, with appropriate adjustments, we can extend it to the general equilibrium problem.

Besides the researches related to equilibrium problems, another class of problems that is also mentioned in this thesis is the fixed point problem. The fixed point theory has been around for about many centuries and has developed strongly in recent decades. Banach contraction mapping principle (1922) has formed two main directions of fixed point theory: The existence of fixed points of continuous mapping and the existence of fixed points of contraction mapping. The Brouwer fixed point principle (1912) and the Banach contraction mapping principle are the initial results for the fixed point theory. In the 60s of the 20th century, the contractile fixed point principle was strongly developed. This theory allows us to build the algorithms to solve the problem.

In recent years, many researchers have been interested in the problem of finding solutions of an equilibrium problem on a set of solutions of other equilibrium problems or finding solutions of an equilibrium prob- lem on a set of common fixed points of mappings. This class of problems is called bi-level equilibrium problems. Along with qualitative research directions, research on methods of solving this problem and applying this problem into practical models plays a very important role.

Realizing the importance and necessity of researching effective algorithms on computers with applications in practical models, the author of this thesis aims to propose new algorithms and applying calculations on Matlab software with specific data.

Beside the introduction, the conclusion, list of published works, and the reference section, the thesis consists of 3 chapters. The main results of the thesis are presented in Chapter 2 and 3.

The main results of the thesis are written based on 04 articles, in which 03 articles were published in journals with ranking SCIE and 01 article was accepted for publication in a journal with ranking SCIE.

# Chapter 1

# Some basic knowledge about equilibrium problems and fixed point problems

In this chapter, we will present some basic concepts as well as necessary supporting results used in the following chapters.

The chapter content is presented in three main parts. The first part recalls the necessary concepts of Functional Analysis and Convex Analysis related to the thesis. The second part introduces the equilibrium problem and its separate cases together the conditions for the existence of solutions to the equilibrium problem. Finally, we present the fixed point problem and some basic iterative methods used to solve this problem.

The chapter content is written based on the documents Cegielski, A. (2013), Konnov, I.V. (2001), Muu, L.D. (2016), Facchinei, F. (2003), Fan, K. (1972), Giannessi, F. (2004).

## 1.1 Hilbert space

## 1.2 Equilibrium problem

### 1.2.1 The problem and some related problems

### 1.2.2 The existence of solutions

## 1.3 The equilibrium problem over the fixed point set

### 1.3.1 The problem

### 1.3.2 Some common algorithms

# Chapter 2

# Extended projection methods

In this chapter, we present two projection methods to solve the equilibrium problem on a set of fixed points in real Hilbert space $\mathbb{H}$ with the assumption that the bifunction $f$ is strongly monotonone and has an approximate sub-differential set is Hausdorff Lipschitz continuous on the set $C$.

The first algorithm is a combination of the approximate sub-gradient technique of Santos P. (2011) and the hybrid reduction direction scheme of Yamada I. (2013). In the second algorithm, we combine the fixed point iteration technique of Mann W.R. (2003) and the parallel sub-gradient method to solve the (2.1) problem when $C = \mathbb{H}$.

Motivated by the extragradient method for equilibrium and fixed point problems in (Anh, P.N., 2013), parallel techniques in Anh P.N. (2020) and Strodiot J.J., Hai T.N. (2022, 2012), we propose a new iter-ative algorithm to solve Problem (2.7), replacing the metric projection as usual by inexact projection. Illustrative calculations of the algorithm and comparison results with other algorithms are also presented in detail in sections 2.3 and 2.4.3. The content of this chapter is written based on two articles [CT1., CT4.] in the The list of works of author related to the Thesis.

Let $S_i : C \to C$ $(i \in I := \{1, 2, \cdots, p\})$ being $\beta_i-$semicontractive mapping, we have an equilibrium problem over the fixed point set:

$$\text{Find } x^* \in \Omega \text{ such that } f(x^*, y) \geq 0, \quad \forall y \in \Omega, \tag{2.1}$$

where $\Omega = \cap_{i\in I} Fix(S_i)$, and $Fix(S_i) := \{x \in C : S_i(x) = x\}$.

## 2.1 Approximately parallel projection method

**Algorithm 2.1.** ***Initialization:*** *Take $x^0 \in C$ arbitrarily.*

***Iterative steps:*** $k = 1, 2, \cdots$

*Step 1. Choose the parameters satisfy the following restrictions:*

$$\begin{cases} \tau \in (0, \beta), 0 < \tau_k \leq \gamma_k < \min\left\{\frac{2\beta}{L^2}, \frac{2(\beta-\tau)}{L^2-\tau^2}, \frac{1}{\tau}\right\}, \\ 0 < a \leq \alpha_{k,i} < \min\left\{\frac{1-\beta_i}{2} : i \in I\right\}, \\ \epsilon_k \leq \gamma_k, \sum_{k=0}^{\infty} \epsilon_k^2 < +\infty, \\ \sum_{k=0}^{\infty} \gamma_k = +\infty, \sum_{k=0}^{\infty} \gamma_k^2 < +\infty, \sum_{k=0}^{\infty} \gamma_k \tau_k < +\infty. \end{cases} \tag{2.2}$$

*Step 2. Compute*

$$\begin{cases} y_i^k = (1-\alpha_{k,i})x^k + \alpha_{k,i} S_i(x^k), \quad \forall i \in I, \\ y^k := y_{i_0}^k, \ \textit{with } i_0 \in \operatorname{argmax}\{\|y_i^k - x^k\| : i \in I\}, \\ x^{k+1} \in Pr_C^{\epsilon_k}(y^k - \gamma_k u^k), u^k \in \partial_2^{\tau_k} f(y^k, y^k). \end{cases} \tag{2.3}$$

*Step 3. Set $k := k+1$ and return to Step 1.*

To prove the convergence of the iterative sequence determined by the algorithm 2.1, we first prove the lemma 2.1.

**Lemma 2.1.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Let $g : C \times C \to \mathcal{R}$ be a bifunction such that $g(x, x) = 0$ for all $x \in C$, and for each $x \in C$, $g(x, y)$ is lower semi-continuous, convex and subdifferentiable on $C$ respect to $y$. For each $\epsilon \geq 0$, if $g$ is $\beta$-strongly monotone on $C$ and $\partial_2^{\epsilon} g(x, x)$ is compact, Lipschitz continuous with constant $L > 0$ on $C$ such that $\beta \leq L$, then the multivalued mapping*

$$S(x) := \{x - \gamma w_x : w_x \in \partial_2^{\epsilon} g(x, x)\}, \quad \forall x \in C,$$

*is* $2\sqrt{\gamma\epsilon}-$*contractive with constant* $\delta = \sqrt{1-\gamma(2\beta-\gamma L^2)}$*, where* $\gamma \in \left(0, \frac{2\beta}{L^2}\right)$.

In addition, we also use some basic lemmas which are restated below.

**Lemma 2.2.** *Let* $\{a_k\}$ *and* $\{\delta_k\}$ *be sequences of nonnegative real numbers such that*

$$a_{k+1} \leq a_k + \delta_k, \quad \forall k \geq 0,$$

*where* $\{\delta_k\}$ *satisfies* $\sum\limits_{k=0}^{\infty} \delta_k < \infty$*. Then, there exists finite the limit* $\lim\limits_{k\to\infty} a_k$.

**Lemma 2.3.** *Let* $\{a_k\}$ *be a sequence of nonnegative real numbers. Suppose that for any integer* $m$*, there exists an integer* $p$ *such that* $p \geq m$ *and* $a_p \leq a_{p+1}$*. Let* $k_0$ *be an integer such that* $a_{k_0} \leq a_{k_0+1}$ *and define, for all integer* $k \geq k_0$*,*

$$\tau(k) = \max\{i \in \mathcal{N} : k_0 \leq i \leq k, a_i \leq a_{i+1}\}.$$

*Then,* $0 \leq a_k \leq a_{\tau(k)+1}$ *for all* $k \geq k_0$*. Furthermore, the sequence* $\{\tau(k)\}_{k\geq k_0}$ *is nondecreasing and tends to* $+\infty$ *as* $k \to \infty$*.*

**Lemma 2.4.** *Assume that* $S : \mathbb{H} \to \mathbb{H}$ *be a* $m-$*demicontractive mapping such that* $Fix(S) \neq \emptyset$ *and* $\alpha \in [0, 1-m]$*. Then, the mapping* $S_\alpha = (1-\alpha)I + \alpha S$ *is quasinonexpansive on* $\mathbb{H}$*. Moreover,*

$$\|S_\alpha(x)-x^*\|^2 \leq \|x-x^*\|^2-\alpha(1-m-\alpha)\|S(x)-x\|^2, \ \ \forall x \in C, x^* \in Fix(S).$$

**Lemma 2.5.** *Let* $\{a_k\} \subset \mathcal{R}_+$ *be a sequence satisfying the inequality*

$$a_{k+1} \leq (1-\alpha_k)a_k + \alpha_k\delta_k$$

*where* $\{\alpha_k\} \subset [0, 1]$ *and* $\{\delta_k\} \subset \mathcal{R}$*. If* $\sum_{k=0}^{\infty} \alpha_k = +\infty$ *and* $\limsup_{k\to\infty} \delta_k \leq 0$*, then* $\lim_{k\to\infty} a_k = 0$*.*

Using lemma 2.1 and the lemmas repeated above, We prove the convergence of the parallel projection method through the theorem 2.1.

**Theorem 2.1.** *Let $f : C \times C \to \mathcal{R} \cup \{+\infty\}$ be $\beta-$strongly monotone and weakly continuous for each $\epsilon \geq 0$ and $x \in C$, $\partial_2^\epsilon f(x, x)$ be compact, Lipschitz continuous with constant $L > 0$ on $C$ such that $\beta \leq L$. For each $i \in I$, let the mapping $S_i : C \to C$ be $\beta_i-$demicontractive such that $\Omega \neq \emptyset$. Then, the sequences $\{x^k\}$ and $\{y^k\}$ converge strongly to a unique solution $x^*$ of the problem* (2.1)*.*

## 2.2 Parallel subgradient method

In this section, we introduce the parallel subgradient projection method to solve the equilibrium problem over a fixed points set and prove the convergence of the algorithm with the assumption that the bifunction $f$ is $\beta-$ strong monotone, weakly continuous for each $\epsilon \geq 0$, $x \in \mathbb{H}$. To prove the convergence of the iterative sequence $\{x^k\}$, $\{y^k\}$ we need to calculate the distance projection onto the set $C$ at each iteration step.

**Algorithm 2.2.** ***Initialization:*** *Take $x^0 \in \mathbb{H}$.*

***Iterative steps:*** $k = 1, 2, \cdots$

$$\begin{cases} \tau \in (0, \beta), 0 < \tau_k \leq \gamma_k < \min\left\{\frac{2\beta}{L^2}, \frac{2(\beta-\tau)}{L^2-\tau^2}, \frac{1}{\tau}\right\}, \\ 0 < a \leq \alpha_{k,i} < \min\left\{\frac{1-\beta_i}{2} : i \in I\right\}, \epsilon_k \leq \gamma_k, \sum_{k=0}^\infty \epsilon_k^2 < +\infty, \\ \sum_{k=0}^\infty \gamma_k = +\infty, \sum_{k=0}^\infty \gamma_k^2 < +\infty, \sum_{k=0}^\infty \gamma_k \tau_k < +\infty, \mu \in \left(0, \frac{2\beta}{L^2}\right), \\ \beta_k \in \left(0, 1 - \gamma_k(1 - \sqrt{1 - 2\mu\beta + \mu^2 L^2})\right). \end{cases} \tag{2.4}$$

*Step 2. Compute*

$$\begin{cases} y_i^k = (1 - \alpha_{k,i})x^k + \alpha_{k,i} S_i(x^k), \quad \forall i \in I, \\ y^k := y_{i_0}^k, \ \textit{with } i_0 \in \operatorname{argmax}\{\|y_i^k - x^k\| : i \in I\}, \\ x^{k+1} = \beta_k x^k + (1 - \beta_k) y^k - \mu\gamma_k u^k, u^k \in \partial_2^{\tau_k} f(y^k, y^k). \end{cases} \tag{2.5}$$

*Step 3. Take $k := k + 1$ and return to Step 1.*

**Theorem 2.2.** *Let the bifunction* $f : \mathbb{H} \times \mathbb{H} \to \mathcal{R} \cup \{+\infty\}$ *be* $\beta-$*strongly monotone and weakly continuous,* $\partial_2^\epsilon f(x, x)$ *be compact, Lipschitz continuous with constant* $L$ *for each* $\epsilon > 0$ *such that* $\beta \leq L$. *For each* $i \in I$*, the mappings* $S_i : \mathbb{H} \to \mathcal{H}$ *be* $\beta_i-$*demicontractive such that* $\Omega \neq \emptyset$. *Then, under the restriction set* (2.4) *onto parameters, the sequences* $\{x^k\}$ *and* $\{y^k\}$ *converge strongly to a unique solution* $x^*$ *of the problem* (2.1).

## 2.3 Numerical illustration

In this section, we will perform some calculations illustrating the strong convergence of iterative sequences generated from the algorithms. We also compare the proposed algorithm with the subgradient type method $((STM))$ of Iiduka and Yamada (2009) (Algorithm 3.2), the extragradient subgradient method of Anh, P.N., Kim, J.K., Muu, L.D. (2012).

### 2.4 Extragradient subgradient parallel projection algorithm

Let $C$ be a nonempty closed convex subset of $\mathbb{H}$. Let $f : C \times C \to \mathcal{R} \cup \{+\infty\}, g_j : C \times C \to \mathcal{R} \cup \{+\infty\}$ be bifunctions such that $f(x,x) = 0, g_j(x,x) = 0$ for all $x \in C, j \in J,\ I = \{1, \cdots, r\}, J = \{1, \cdots, m\}$. The equilibrium problem for $f$ onto $C$: Find $\bar{x} \in C$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in C. \tag{2.6}$$

The solution set of the problem is denoted by $S(C, f)$. Let mappings $S_i : C \to C (i \in I)$ be demicontractive. In this paper, we consider the following bilevel equilibrium problem including demicontractive mappings:

$$\text{Find } x^* \in \Omega \ \text{ such that } f(x^*, y) \geq 0, \quad \forall y \in \Omega, \tag{2.7}$$

where $Fix(S_i)$ is the fixed point set of $S_i$, and $\Omega = \cap_{i \in I} Fix(S_i) \cap S(C, g_j),\ j \in J$.

The convergence of the sequences generated by the algorithms is proven in Theorem 2.3. Calculation results in infinite as well as finite dimensional space are given to illustrate the calculation of the convergence of the sequences generated by the algorithm.

#### 2.4.1 Algorithm and the convergence

In this section, we will present the calculation steps of the algorithm with approximation subgradient technique and approximation of distance projection.

**Algorithm 2.3.** ***Initialization:*** *Choose any* $x^0 \in C$.

*Step 1. Choose the parameters*

$$\begin{cases} \tau \in (0,\beta), \tau_k \le \gamma_k < \min\left\{\frac{2\beta}{L^2}, \frac{2(\beta-\tau)}{L^2-\tau^2}, \frac{1}{\tau}\right\}, \\ 0 < a \le \alpha_{k,i} \le \min\left\{\frac{1-\beta_i}{2} : i \in I\right\}, \\ 0 < \bar{a} \le \rho_{k,j} \le \bar{b} < \min\left\{\frac{1}{2c_{1j}}, \frac{1}{2c_{2j}} : j \in J\right\}, \\ \epsilon_k \le \gamma_k, \sum_{k=0}^{\infty} \epsilon_k^2 < +\infty, \\ \sum_{k=0}^{\infty} \gamma_k = +\infty, \sum_{k=0}^{\infty} \gamma_k^2 < +\infty, \sum_{k=0}^{\infty} \gamma_k \tau_k < +\infty. \end{cases} \tag{2.8}$$

*Step 2. Compute*

$$\begin{cases} y_i^k = (1-\alpha_{k,i})x^k + \alpha_{k,i} S_i(x^k), \quad \forall i \in I, \\ y^k := y_{i_0}^k, \;\; \mathit{with} i_0 \in \operatorname{argmax}\{\|y_i^k - x^k\| : i \in I\}, \\ z_j^k = \operatorname{argmin}\left\{\rho_{k,j} g_j(y^k, y) + \frac{1}{2}\|y - y^k\|^2 : y \in C\right\}, \\ \bar{z}_j^k = \operatorname{argmin}\left\{\rho_{k,j} g_j(z_j^k, y) + \frac{1}{2}\|y - y^k\|^2 : y \in C\right\}, \\ z^k := \bar{z}_{j_0}^k, \;\; \mathit{with}\ j_0 \in \operatorname{argmax}\{\|\bar{z}_j^k - y^k\| : j \in J\}, \\ x^{k+1} \in Pr_C^{\epsilon_k}(z^k - \gamma_k u^k), u^k \in \partial_2^{\tau_k} f(z^k, z^k). \end{cases} \tag{2.9}$$

*Step 3. Take $k := k+1$ and return to Step 1.*

Now we will discuss the iteration scheme and the convergence of the parallel projection method with computing inexact subgradients and approximate metric projections. Consider this Lemmas.

**Lemma 2.6.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Let $g : C \times C \to \mathcal{R}$ be a bifunction such that $g(x,x) = 0$ for all $x \in C$, and for each $x \in C$, $g(x,y)$ is lower semi-continuous, convex and subdifferentiable on $C$ respect to $y$. For each $\epsilon \ge 0$, if $g$ is $\beta$-strongly monotone on $C$ and $\partial_2^{\epsilon} g(x,x)$ is Lipschitz continuous with constant $L > 0$ on $C$, then the multivalued mapping*

$$S(x) := \{x - \gamma w_x : w_x \in \partial_2^{\epsilon} g(x,x)\}, \quad \forall x \in C,$$

*is $2\sqrt{\gamma\epsilon}-$contractive with constant $\delta = \sqrt{1-\gamma(2\beta-\gamma L^2)}$, where $\gamma \in \left(0, \frac{2\beta}{L^2}\right)$.*

**Lemma 2.7.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$, and a bifunction $h : C \times C \to \mathcal{R} \cup \{+\infty\}$ satisfy the conditions:*

- *$h(x, x) = 0$ for all $x \in C$;*
- *for each $x \in C$, $h(x, \cdot)$ is convex and subdifferentable on $C$;*
- *$h$ is pseudomonotone on $C$;*
- *$h$ is Lipschitz-type with constants $\gamma_1 > 0$ and $\gamma_2 > 0$.*

*Then, if $\lambda \in \left(0, \min\left\{\frac{1}{2\gamma_1}, \frac{1}{2\gamma_2}\right\}\right)$, then the mapping $S$ is defined in, for each $x \in C$,*

$$y_x = \operatorname{argmin}\left\{\lambda h(x, y) + \frac{1}{2}\|y - x\|^2 : y \in C\right\},$$
$$S(x) = \operatorname{argmin}\left\{\lambda h(y_x, y) + \frac{1}{2}\|y - x\|^2 : y \in C\right\},$$

*which is quasinonexpansive on $C$.*

Now we will discuss the iteration scheme and the convergence of the parallel projection method with computing inexact subgradients and approximate metric projections.

**Theorem 2.3.** *Let $f$ be $\beta-$strongly monotone and weakly continuous, $\partial_2^\epsilon f(x, x)$ be $L-$ Lipschitz continuous on $C$. For each $i \in I$, let the mapping $S_i : C \to C$ be $\beta_i-$demicontractive such that $\Omega \neq \emptyset$. Let $g_j (j \in J)$ be pseudomonotone, weakly continuous and Lipschitz-type with constants $c_{1j}$ and $c_{2j}$. Then, the sequences $\{x^k\}, \{y^k\}$ and $\{z^k\}$ converge strongly to a unique solution $x^*$ of Problem* (2.1).

In this section, we suppose that $f, S_i (i \in I)$ and $g : C \times C \to \mathcal{R} \cup \{+\infty\}$ satisfy the following assumptions:

(1) The bifunction $f$ is $\beta-$strongly monotone. weakly continuous and $\partial_2^\epsilon f(x, x)$ is Lipschitz continuous on $C$ with constant $L > 0$ for all $\epsilon > 0$;

(2) The mappings $\{S_i : i \in I\}$ are $\beta_i-$demicontractive;

(3) The bifunction $g$ is pseudomonotone, weakly continuous, Lipschitz-type with constants $c_1 > 0$ and $c_2 > 0$, $g(x,x) = 0$ for all $x \in C$.

When $S_i (i \in I)$ is the identity mapping and $g_j = g (j \in J)$, we give the following application of Theorem 2.3.

**Corollary 2.1.** *Let positive paramerter sequences* $\{\rho_k\}, \{\epsilon_k\}, \{\gamma_k\}$ *and* $\{\tau_k\}$ *satisfy the restriction set:*

$$\begin{cases} \tau \in (0,\beta), 0 < \tau_k \le \gamma_k < \min\left\{\frac{2\beta}{L^2}, \frac{2(\beta-\tau)}{L^2-\tau^2}, \frac{1}{\tau}\right\}, \\ 0 < \bar{a} \le \rho_k \le \bar{b} < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}, \\ \epsilon_k \le \gamma_k, \sum_{k=0}^{\infty} \epsilon_k^2 < +\infty, \\ \sum_{k=0}^{\infty} \gamma_k = +\infty, \sum_{k=0}^{\infty} \gamma_k^2 < +\infty, \sum_{k=0}^{\infty} \gamma_k \tau_k < +\infty. \end{cases}$$

*Then, the sequences* $\{x^k\}$ *and* $\{y^k\}$ *are defined by the iteration scheme:*

$$\begin{cases} x^0 \in C, \\ y^k = \text{argmin}\left\{\rho_k g(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}, \\ z^k = \text{argmin}\left\{\rho_k g(y^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}, \\ x^{k+1} \in Pr_C^{\epsilon_k}(z^k - \gamma_k u^k), u^k \in \partial_2^{\tau_k} f(z^k, z^k), \end{cases} \tag{2.10}$$

*which converge strongly to a unique solution of the bilevel equilibrium problem* (2.7).

In the case $g_j = 0 (j \in J)$, Problem (2.1) is formulated in the equilibrium problem over the fixed point set of the demicontractive mappings $S_i (i \in I)$. By Theorem 2.3, the iteration scheme for solving Problem (2.7) and its convergence are given as the following results.

**Corollary 2.2.** *Suppose that the sequences* $\{x^k\}$ *and* $\{z^k\}$ *generated* *by the scheme:*

$$\begin{cases} x^0 \in C, \\ y_i^k = (1-\alpha_{k,i})x^k + \alpha_{k,i}S_i(x^k), \quad \forall i \in I, \\ y^k := y_{i_0}^k, \ \textit{where } i_0 \in \operatorname{argmax}\{\|y_i^k - x^k\| : i \in I\}, \\ x^{k+1} \in Pr_C^{\epsilon_k}(y^k - \gamma_k u^k), u^k \in \partial_2^{\tau_k} f(y^k, y^k). \end{cases} \tag{2.11}$$

*Choosing positive paramerter sequences* $\{\alpha_{k,i}\}(i \in I), \{\epsilon_k\}, \{\gamma_k\}$ *and* $\{\tau_k\}$ *satisfies the conditions:*

$$\begin{cases} \tau \in (0,\beta), 0 < \tau_k \le \gamma_k < \min\left\{\frac{2\beta}{L^2}, \frac{2(\beta-\tau)}{L^2-\tau^2}, \frac{1}{\tau}\right\}, \\ 0 < a \le \alpha_{k,i} \le \min\left\{\frac{1-\beta_i}{2} : i \in I\right\}, \\ \epsilon_k \le \gamma_k, \sum_{k=0}^{\infty} \epsilon_k^2 < +\infty, \\ \sum_{k=0}^{\infty} \gamma_k = +\infty, \sum_{k=0}^{\infty} \gamma_k^2 < +\infty, \sum_{k=0}^{\infty} \gamma_k \tau_k < +\infty. \end{cases}$$

*Then, the sequences* $\{x^k\}$ *and* $\{y^k\}$ *converge strongly to a unique solution* $x^*$ *of Problem* (2.7).

### 2.4.2 Numerical illustration

In this section, we present some numerical calculations illustrating the calculation steps of the Algorithms. The calculations are perform in MATLAB R2014a on PC Intel(R) Core(TM) i5-7360U CPU @ 2.30GHz 8.00GB Ram. We also compare the convergence of the sequences generated by the proposed calculation scheme (2.11) with the subgradient type method given by Iiduka H., Yamada I. (2009) (algorithm 3.2), Scheme (2.3) and the approximate and contactive algorithm in Hai, T.N. (2017) (algorithm 4.1), Scheme (2.10) and Proximal subgradient of Anh P.N. (2017) (algorithm 2).

# Chapter 3

# Inertial subgradient methods

In this chapter, we introduce new iterative algorithms to solve the equilibrium problem where the constrained sets are given as the intersection of the fixed point sets of demicontractive mappings in a real Hilbert space.

The first algorithm uses a new technique, the hybrid reduction method, the subgradient technique to solve the equilibrium problem $EQ(\Omega, f)$ with the assumption that the bifunction $f$ is strongly monotonone and type Lipschitz continuous on $\mathbb{H}$. The second algorithm is based on inertial extrapolation, parallel and auxiliary principle techniques. The strong convergence of the iterative sequence generated by the algorithms is proven in Theorems 3.1 and Theorem 3.2 with appropriate parameters.

## 3.1 Inertial subgradient methods

### 3.1.1 Algorithm and the convergences

For solving the equilibrium problems over the fixed point set the EPF$(\Omega, f)$, we assume the bifunction $f$ and the mappings $S_k (k \in I)$, parameters satisfy the following conditions:

$(A_1)$ The $f$ is $\beta$-strongly monotone, the subdifferential $\partial_2 f(x,x)$ is compact and $L$-Lipschitz continuous;

$(A_2)$ For each $k \in I$, $S_k$ is $\xi_k-$demicontractive and satisfies the condition $(Z)$, $\Omega := \bigcap_{k\in I} \mathrm{Fix}(S_k) \neq \emptyset$;

$(A_3)$ For every $k \geq 0$, positive parameters $\beta_k, \gamma_k, \tau_k, \lambda_k$ and $\{\mu_k\}$ satisfy the following restrictions:

$$\begin{cases} 0 < c_1 \leq \beta_k \leq c_2 < 1, \mu_k \leq \eta, \\ \alpha_k \in (0, 1-\xi_k], \inf_k \alpha_k > 0, \\ 0 < \gamma_k < 1, \lim\limits_{k\to\infty} \gamma_k = 0, \sum_{k=1}^{\infty} \gamma_k = \infty, \\ \lim\limits_{k\to\infty} \frac{\tau_k}{\gamma_k} = 0, \lambda_k \in \left(\frac{\beta}{L^2}, \frac{2\beta}{L^2}\right), a \in (0,1), \sqrt{1-2\lambda_k\beta+\lambda_k^2 L^2} < 1-a. \end{cases} \tag{3.1}$$

**Algorithm 3.1. (Inertial hybrid subgradient algorithm)**

***Initialization:*** *Take $x^0, x^1 \in \mathcal{H}$ arbitrarily.*

***Iterative steps:*** $k = 1, 2, \ldots$

*Step 1. Compute an inertial parameter*

$$\theta_k = \begin{cases} \min\left\{\mu_k, \dfrac{\tau_k}{\|x^k - x^{k-1}\|}\right\} & \text{if } \|x^k - x^{k-1}\| \neq 0, \\ \mu_k & \text{otherwise.} \end{cases} \tag{3.2}$$

*Step 2. Compute*

$$\begin{cases} w^k = x^k + \theta_k(x^k - x^{k-1}) \quad (\textit{inertial technique}), \\ \bar{S}_k w^k = (1-\alpha_k)w^k + \alpha_k S_k w^k, \\ y^k \in \partial_2 f(w^k, w^k) \quad (\textit{compute subradient}), \\ z^k = (1-\gamma_k)\bar{S}_k w^k + \gamma_k \left[w^k - \lambda_k y^k\right], \\ \bar{S}_k z^k = (1-\alpha_k)z^k + \alpha_k S_k z^k, \\ x^{k+1} = (1-\beta_k)\bar{S}_k w^k + \beta_k \bar{S}_k z^k. \end{cases} \tag{3.3}$$

*Step 3. Set $k := k+1$ and return to Step 1.*

**Lemma 3.1.** *Let $\{s_k\}$ be a sequence of nonnegative real numbers and $\{p_k\}$ a sequence of real numbers. Let $\{\alpha_k\}$ be a sequence of real numbers in $(0,1)$ such that $\sum_{k=1}^{\infty} \alpha_k = \infty$. Assume that*

$$s_{k+1} \leq (1-\alpha_k)s_k + \alpha_k p_k, \quad k \in \mathcal{N}.$$

*If* $\limsup_{i\to\infty} p_{k_i} \le 0$ *for every subsequence* $\{s_{k_i}\}$ *of* $\{s_k\}$ *satisfying*

$$\liminf_{i\to\infty}(s_{k_i+1} - s_{k_i}) \ge 0,$$

*then* $\lim_{k\to\infty} s_k = 0$.

A strong convergence result is established by the following theorem.

**Theorem 3.1.** *Assume that the assumptions* $(A_1) - (A_3)$ *are satisfied. Then, the sequence* $\{x^k\}$ *generated by the algorithm 3.1 converges strongly to a unique solution* $x^*$ *of the problem* $\mathrm{EQ}(\Omega, f)$.

#### 3.1.2 Numerical illustration

In this section, we will perform some calculations illustrating the strong convergence of the sequences generated by the algorithm. We also compare the proposed algorithm with the parallel projection method ($PPA$) (Anh .P.N., 2022), Algorithm 3.1, Algorithm 4.1.

### 3.2 Parallel inertial auxiliary principle technique

#### 3.2.1 Algorithm and the convergence

We assume that:

($A_1$) The mapping $f : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ is $\beta-$strongly monotone and Lipschitz continuous with positive constants $c_1, c_2$ such that $\beta > c_1$.

($A_2$) For all $i \in I$ the mappings $S_i : \mathcal{H} \to \mathcal{H}$ are $\beta_i-$demicontractive and demiclosed at zero and the set $\Omega := \cap_{i\in I} Fix(S_i)$ is nonempty.

The parameters setup for the algorithm is as follows.

$$\begin{cases} \tau \in (0, \beta - c_1), \{\lambda_k\} \subset [\bar{a}, \hat{a}] \subset (0,1), \lambda_k^2 + \frac{\tau - 4(\beta - c_1)}{2\tau^2(\beta - c_1)}\lambda_k + \frac{\beta - c_1 - \tau}{\tau^2(\beta - c_1)} \ge 0, \\ \zeta_k \in (0, \frac{1}{\tau\bar{a}}), \sum\limits_{k=1}^{\infty} \zeta_k = +\infty, \tau_k > 0, \sum\limits_{k=1}^{\infty} \tau_k < +\infty, \\ \mu_k > 0, \gamma_{k,i} \in (\bar{b}, \hat{b}) \subset (0, 1 - \max\{\beta_i : i \in I\}), \quad \forall i \in I. \end{cases} \tag{3.4}$$

**Algorithm 3.2.** *Choose starting points $x^0, x^1 \in \mathcal{H}$.*

*Step 1. (Inertial technique) Given the iterates $x^{k-1}$ and $x^k$, compute*

$$w^k = x^k + \alpha_k(x^k - x^{k-1}), \tag{3.5}$$

*where*

$$\alpha_k = \begin{cases} \min\left\{\frac{\tau_k}{\|x^k - x^{k-1}\|}, \mu_k\right\}, & if \quad \|x^k - x^{k-1}\| \neq 0, \\ \mu_k & otherwise. \end{cases} \tag{3.6}$$

*Step 2. (Parallel technique) Take*

$$u_i^k = (1 - \gamma_{k,i})w^k + \gamma_{k,i}S_i(w^k).$$

*Set $t^k := u_{i_0}^k$, where $i_0 \in \operatorname{argmax}\{\|u_i^k - w^k\| : i \in I\}$.*

*Step 3. (Auxiliary problem principle) Compute*

$$y^k = argmin\left\{\lambda_k f(t^k, x) + \frac{1}{2}\|x - t^k\|^2 : x \in C\right\},$$

*$x^{k+1} = (1 - \zeta_k)t^k + \zeta_k y^k$. Let $k := k + 1$ and go to Step 1.*

Note that, computing $w^k$ is used by inertial technique and $t^k$ is by parallel technique. Then, the iteration point $x^{k+1}$ is based on the Mann iteration method and the auxiliary problem principle. We recall that a point $x^k$ generated by Algorithm 3.2 is an $\epsilon-$solution of the problem $EQ(\Omega, f)$, if $\|x^{k+1} - x^k\| \leq \epsilon$.

For the convergence of the algorithm we assume the following Lemma.

**Lemma 3.2.** *Let $\{a_k\}$ is positive and the sequence $\{p_k\}$. Take a real sequence $\{\alpha_k\}$ in $(0, 1)$ such that $\sum\limits_{k=1}^{\infty} \alpha_k = \infty$. Assume*

$$a_{k+1} \leq (1 - \alpha_k)a_k + b_k, \quad k = 1, 2, \cdots.$$

*Then, if $\limsup\limits_{k\to\infty} \frac{b_k}{\alpha_k} \leq 0$ or $\sum\limits_{k=1}^{\infty} b_k < +\infty$, then $\lim\limits_{k\to\infty} a_k = 0$.*

**Theorem 3.2.** *Assume that Assumptions $(A_1)$ and $(A_2)$ hold. Under conditions (3.4), the sequence $\{x^k\}$ generated by Algorithm 3.2 strongly converges to a unique solution $x^*$ of the problem $EQ(\Omega, f)$.*

### 3.2.2 Numerical illustration

In this section, we will do some numerical calculations. The proposed algorithm (PIAPA) will be compared with the Parallel Projection Algorithm ($PPA$) (CT1., Scheme 3.1) and subgradient type algorithm ($STA$) by Iiduka H. (2003) (Algorithm 3.2).

# Conclusions

**1. 1. Research results:**

In this thesis, we have built a number of algorithms to solve the equilibrium problems over the fixed point sets. The thesis has achieved the following results.

- Proposing two new algorithms to solve the equilibrium problem over a fixed point set in a real Hilbert space $\mathbb{H}$.This result is published in [CT1.], List of published works of the author.
- Proposing a new projection method to solve equilibrium problems defined over the intersection of fixed point sets and the solution of the equilibrium problem. The convergence of repeating sequences generated by the algorithm is proven Theorem 2.3. This result is published in [CT4.], List of published works of the author.
- Proposing two algorithms to solve the equilibrium problems over the intersection of fixed point sets of demicontractive mappings in Hilbert space. Under the assumption that the bifunction $f$ is strongly monotone and Lipschitz type continuous on $\mathbb{H}$, the sequences generated by the two algorithms all converge strongly to the solution of the problem. This result is published in [CT2., CT3.], List of published works of the author.
- Performing Several numerical experiments in finite and infinite dimensional spaces with comparisons to related results to illustrate the algorithm performances and emphasize its computational and convergence advantages.

**2. Recommendations for further studies:** Future research that can extend our current study:

- Research new algorithms for solving bilevel equilibrium problems in particular and general;
- Evaluate the error and convergence speed of the algorithms proposed in the thesis;
- Apply proposed algorithms to practical models and calculate algorithm complexity.

## The list of works of author related to the Thesis

CT1. Anh, P.N., Hong, N.V., *New projection methods for equilibrium problems over fixed point sets.* Optimization Letters, 2021, **15** (2), 627-648 (ISSN: 1862-4472, SCIE, Q1).

CT2. Anh, P.N., Kim, J.K., Hien, N.D., Hong, N.V., *Strong convergence of inertial hybrid subgradient methods for solving equilibrium problems in Hilbert spaces.* Journal of Nonlinear and Convex Analysis, 2023, **24** (3), 499-514 (ISSN: 1345-4773, SCIE, Q2).

CT3. Hien, N.D., Hong, N.V., Kim, J.K., *Auxiliary problem principle extended to equilibrium problems over the intersection of fixed point sets.* Accepted by Journal of Nonlinear and Convex Analysis, 2023 (ISSN: 1345-4773, SCIE, Q2).

CT4. Anh, P.N., Hong, N.V., Gibali, A., *Inexact simultaneous projection method for solving bilevel equilibrium problems.* Fixed Point Theory, 2023, **24** (2), 487-506 (ISSN: 1583-5022, SCIE, Q2).